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Second Hankel Determinant for Certain Class of Functions Defined by Differential Operator

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Abstract:
The objective of this paper is to obtain an upper bound of second Hankel determinant for a class of functions $M^k_{\alpha,\beta,\lambda,\delta}$ (φ) defined in the open unit disc U by using the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$. In addition, we give particular values to the parameters A, B and k to study special cases of the results of this article. The class $M^k_{\alpha,\beta,\lambda,\delta}$ (φ) and the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$.

Key words: differential operator, Second Hankel determinant, Starlike functions, Subordination property.

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محدد هنكل الثاني لفئة معينة من الدوال المعرفة بواسطة مؤثر تفاضلي

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الملخص :

الهدف من هذه الورقة هو الحصـــول على الحد الأعلى من محدد هنكل الثاني لفئة من الدوال $M^{k}_{\alpha,\beta,\lambda,\delta}$ (φ) المعرفة في القرص الأحادي المفتوح U بواسـطة اسـتخدام *المؤثر التفاضـلي* $D^{k}_{\alpha,\beta,\lambda,\delta}f(z)$ ، ايضاً نعطي قيم خاصة للبارمترات $A, B\ , k$ لدراسة حالات خاصة من نتائج هذا البحث ، فئة الدوال (φ) $M^{k}_{\alpha,\beta,\lambda,\delta}$ ، والمؤثر التفاضلي $D^{k}_{\alpha,\beta,\lambda,\delta}f(z)$.

الكلمات المفتاحية: العامل التفاضـــلي، محدد هنكل الثاني، الدوال النجمية، خاصية التبعية.

1. Introduction:

In 1976, Noonan and Thomas defined the q^{ht} Hankel determinant of *f*(*z*) for $q, n \in N$ as **[1]**

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{\ n+1} & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ a_{n+q-1} & \cdots & a_{n+2q-2} \end{vmatrix}$$

That is, for the complex sequence , $a_n, a_{n+1}, a_{n+2}, \dots$ the Hankel matrix is the infinite matrix whose $(i,j)^{th}$ entry a_{ij} is defined by

$$a_{ij} = a_{n+i+j-2}, \quad (i,j,n \in N).$$

This determinant was discussed by several authors particularly for the cases when $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2,$ that is $H_2(1) = |\, a_3 - a_2^2|$ and $H_2(2) = |\, a_2 a_4 - a_3^2|$

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where $H_2(1)$ is known as the Fekete-Szeqö problem and $H_2(2)$ refer to the second Hankel determinant. For example, and many others have obtained sharp upper bounds of $H_2(2)$ for different classes of analytic functions. **[2][3][4]**

In the present paper, and by making use of the differential operator $D^k_{\alpha,\beta,\lambda,\delta}f(z)$, we will obtain the upper bound of the second Hankel determinant for the class $M^k_{\alpha,\beta,\lambda,\delta}$ (φ) that will be defined below .

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

Defined in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Ramadan and Darus defined the following differential operator for a function $f \in A$ as follows:

$$D^k_{\alpha,\beta,\lambda,\delta}f(z) = +\sum_{n=2}^{\infty}[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k a_n z^n, \tag{2}$$

Where $\alpha,\beta,\lambda,\delta \geq 0,\ \lambda > \delta,\ \ \beta > \alpha$ and $k \in \{0,1,2,\dots\}$. Also in the same paper, they defined the class $M^k_{\alpha,\beta,\lambda,\delta}$ (φ) as follows:

1. Definition [5], Let φ (z) be a univalent starlike function with respect to 1 which maps the unit disk U onto a region in the right half plane which is symmetric with respect to the real axis, φ (0) =1 and $\varphi'(0) > 0$. A function $f \in A$ is in the class $M^k_{\alpha,\beta,\lambda,\delta}$ (φ) if

$$\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}f(z)} < \varphi(z). \tag{3}$$

If J is the class of the functions $G(z)$ defined as

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$$G(z) = \big(1-(\lambda-\delta)(\beta-\alpha)\big)f(z) + (\lambda-\delta)(\beta-\alpha)zf'(z)$$
$$= z + \sum_{n=2}^{\infty}[(\lambda-\delta)(\beta-\alpha)(n-1)+1]^k a_n z^n,$$

For $\alpha,\beta,\lambda,\delta \geq 0$, $\lambda > \delta$, $\beta > \alpha$ and $k \in \{0,1,2,\dots\}$. Then $G(z)$ can be considered as the analytic function in U. Also, let S be defined as the class of functions $G(z) \in J$, which is univalent in U. Using Schwarzian functions $w(z) = \sum_{n=1}^{\infty} d_n z^n$, which are analytic in U and satisfying the conditions $w(0)=0$ and $|w(z)| < 1$. Let f and g be two analytic functions in U. Then, f is a subordinate to g ($f \prec g$) if $f(z)=g(w(z))$ is satisfied.
Now, if we set

$\varphi(z) = \frac{1+Az}{1+Bz} = 1+(A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + \cdots$,
$(-1 \leq B < A \leq 1)$ in (3) we can write that

$$\frac{z\left(D^k_{\alpha,\beta,\lambda,\delta}f(z)\right)'}{D^k_{\alpha,\beta,\lambda,\delta}f(z)} \prec \frac{1+Az}{1+Bz}, \qquad (4)$$

And we can write the class $M^k_{\alpha,\beta,\lambda,\delta}(\varphi)$ as $M^k_{\alpha,\beta,\lambda,\delta}(A,B)$.
Let $M^k_{\alpha,\beta,\lambda,\delta}(A,B)$ be a subclass of the functions $G(z) \in J$ and satisfy the condition

$$\frac{zG'(z)}{G(z)} \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1), \qquad (5)$$

Where $M^k_{\alpha,\beta,\lambda,\delta}(A,B)$ is a subclass of starlike functions and $M^0_{\alpha,\beta,\lambda,\delta}(A,B) \equiv S^*(A,B)$ and $M^0_{\alpha,\beta,\lambda,\delta}(1,-1) \equiv S^*$.

The class S^* is the class of starlike functions and studied by Goel and Mehrok **[6]**.

2. Preliminary Results

Let P denote the class of the functions p which are analytic in U, for which $\text{Re}\{p(z)\} > 0$,

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots = \left[1 + \sum_{n=1}^{\infty} p_n z^n\right], z \in U. \quad (6)$$

In order to investigate our mean result, we need the following lemmas:

2.1. Lemma. **[7]** If $p \in P$, then $|p_k| \leq 2, \ (k = 1,2,3, \dots)$.

2.2. Lemma. [**8**][**9**] If $p \in P$, then

$2p_2 = p_1^2 + (4 - p_1^2)x,$

$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$

for some x and z satisfying $|x| \leq 1, |z| \leq 1 and \ p_1 \in [0,2]$.

3. Main Results

3.1.Theorem. If $G(z) \in M^k_{\alpha,\beta,\lambda,\delta}\ (A, B)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(A - B)^2}{4[2(\lambda - \delta)(\beta - \alpha) + 1]^{2k}} \quad (7)$$

Proof. If $G(z) \in M^k_{\alpha,\beta,\lambda,\delta}\ (A, B)$, then there exists a Schwarz function $w(z)$ such that

$$\frac{zG'(z)}{G(z)} = \varphi\big(w(z)\big), \quad (z \in U), \quad (8)$$

Where

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$$\varphi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-B(A-B)z^2+B^2(A-B)z^3+\cdots$$
$$=1+E_1z+E_2z^2+E_3z^3+\cdots\cdot \tag{9}$$

Furthermore, the function $p_1(z)$ can be defined as follows:

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+b_1z+b_2z^2+b_3z^3+\cdots\cdot \tag{10}$$

Now, we can write $R(p_1(z))>0$ and $p_1(0)=1$. After that,
We define the function $h(z)$ by

$$h(z)=\frac{zG'(z)}{G(z)}=1+c_1z+c_2z^2+c_3z^3+\cdots\cdot \tag{11}$$

From the equations (8), (10) and (11) we have
$$h(z)=\varphi\left(\frac{p(z)-1}{p_1(z)+1}\right)=\varphi\left(\frac{b_1z+b_2z^2+b_3z^3+\cdots}{2+b_1z+b_2z^2+b_3z^3+\cdots}\right)$$
$$=\varphi\left[\frac{1}{2}b_1z+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)z^2+\frac{1}{2}\left(b_3-b_1b_2-\frac{b_1^3}{4}\right)z^3+\cdots\right]$$
$$=1+\frac{E_1b_1}{2}z+\left[\frac{E_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_2b_1^2}{4}\right]z^2+\left[\frac{E_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{E_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_3b_1^3}{8}\right]z^3+\cdots.$$

Thus,
$$c_1=\frac{E_1b_1}{2},\quad c_2=\frac{E_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_2b_1^2}{4},$$

And
$$c_3=\frac{E_1}{2}\left(b_3-b_1b_2+\frac{b_1^3}{4}\right)+\frac{E_2b_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_3b_1^3}{8} \tag{12}$$

Now, by employing (9) and (11) in (12) we get
$[(\lambda-\delta)(\beta-\alpha)+1]^k a_2 = c_1$

Then,

$$a_2 = \frac{c_1}{[(\lambda-\delta)(\beta-\alpha)+1]^k} = \frac{E_1 b_1}{2[(\lambda-\delta)(\beta-\alpha)+1]^k}$$
$$= \frac{(A-B)b_1}{2[(\lambda-\delta)(\beta-\alpha)+1]^k} \qquad (13)$$

After that,

$2[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3 = c_2 + c_1^2 \cdot$
Then,

$$a_3 = \frac{1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}[c_2+c_1^2]$$
$$= \frac{1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left[\frac{E_1}{2}\left(b_2-\frac{b_1^2}{2}\right)+\frac{E_2 b_1^2}{4}+\frac{E_1^2 b_1^2}{4}\right]$$
$$= \frac{1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left[\frac{b_2 E_1}{2}-\frac{E_1 b_1^2}{4}+\frac{E_2 b_1^2}{4}+\frac{E_1^2 b_1^2}{4}\right]$$
$$= \frac{1}{2[2(\lambda-\delta)(\beta-\alpha)+1]^k}\left[\frac{(A-B)b_2}{2}-\frac{(A-B)b_1^2}{4} - \frac{B(A-B)b_1^2}{4}+\frac{(A-B)^2 b_1^2}{4}\right]$$
$$= \frac{A-B}{8[2(\lambda-\delta)(\beta-\alpha)+1]^k}[2b_2-b_1^2-Bb_1^2+(A-B)b_1^2]$$
$$= \frac{A-B}{8[2(\lambda-\delta)(\beta-\alpha)+1]^k}[2b_2-b_1^2-2Bb_1^2+Ab_1^2]$$
$$= \frac{A-B}{8[2(\lambda-\delta)(\beta-\alpha)+1]^k}[2b_2+(A-2B-1)b_1^2], \qquad (14)$$

And

$$4[3(\lambda-\delta)(\beta-\alpha)+1]^k a_4 = c_3 + c_2[(\lambda-\delta)(\beta-\alpha)+1]^k a_2$$
$$+c_1[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3 + [3(\lambda-\delta)(\beta-\alpha)+1]^k a_4,$$
$$3[3(\lambda-\delta)(\beta-\alpha)+1]^k a_4 = c_3 + c_2[(\lambda-\delta)(\beta-\alpha)+1]^k a_2$$
$$+c_1[2(\lambda-\ \delta)\ (\beta-\alpha)+1]^k a_3$$
$$= c_3 + c_2c_1 + c_1[2(\lambda-\delta)(\beta-\alpha)+1]^k a_3, = c_3 + \frac{3c_1c_2}{2} + \frac{c_1^3}{2},$$
$$a_4 = \frac{c_3}{3[3(\lambda-\delta)(\beta-\alpha)+1]^k} + \frac{c_1c_2}{2[3(\lambda-\delta)(\beta-\alpha)+1]^k} + \frac{c_1^3}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}$$
$$= \frac{1}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}[2c_3 + 3c_1c_2 + c_1^3]$$

So that,

$$a_4 = \frac{1}{6[3(\lambda-\delta)(\beta-\alpha)+1]^k}\left[E_1\left(b_3 - b_1b_2 + \frac{b_1^3}{4}\right) + E_2b_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_3b_1^3}{4} + \frac{3E_1b_1}{2}\left(\frac{E_1}{2}\left(b_2 - \frac{b_1^2}{2}\right) + \frac{E_2b_1^2}{4}\right) + \frac{E_1^3b_1^3}{8}\right]$$

$$= \frac{(A-B)[8b_3 + (6A-14B-8)b_1b_2 + (A^2+6B^2-5AB-3A+7B+2)\ b_1^3]}{48[3(\lambda-\delta)(\beta-\alpha)+1]^k}. \quad (15)$$

From (13), (14) and (15) we find that

$$a_2a_4 - a_3^2 = \frac{(A-B)^2}{192} \cdot \left[\frac{R(b_1, b_2, b_3, A, B, \lambda, \delta, \beta, \alpha)}{[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}\right] \quad (16)$$

Where

$$
\begin{aligned}
&R(b_1, b_2, b_3, A, B, \lambda, \delta, \beta, \alpha) \\
&= 16[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} b_1 b_3 \\
&-12[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k b_2^2 \\
&\quad +(12A[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad -12A[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k - 28B[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad +24B[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k - 16[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad +12[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k) b_2 b_1^2 \\
&\quad +(2A^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad -3A^2[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k + 12B^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad -12B^2[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k - 10AB[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad +12AB[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k - 6A[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad +6A[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k + 14B[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad -12B[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k + 4[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} \\
&\qquad -3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha) \\
&\qquad +1]^k) b_1^4.
\end{aligned}
$$

If lemma (2.1) and lemma (2.2) are applied to (16) we have

$$
|a_2 a_4 - a_3^2| = \frac{(A-B)^2 |T(A, B, \lambda, \delta, \beta, \alpha, b_1, x, z)|}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}
$$

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Such that

$$T(A,B,\lambda,\delta,\beta,\alpha,b_1,x,z)=(2A^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-3A^2[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k+12B^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-12B^2[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k-10AB[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}+12AB[(\lambda-\delta)(\beta-\alpha)+1]^k\ [3(\lambda-\delta)(\beta-\alpha)+1]^k)b_1^4+(6A[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-6A[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k-14B[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}+12B[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k)b_1^2(4-b_1^2)x+8[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}b_1(4-b_1^2)(1-|x|^2)z-(4-b_1^2)x^2(4b_1^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}+3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k(4-b_1^2)).$$

By assuming that $b_1=b$ and $b\in[0,2]$, with triangular inequality and $|z|\le 1$ then

$$|a_2a_4-a_3^2|\le\frac{(A-B)^2F(\gamma)}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}$$

Is also obtainable where $\gamma=|x|\le 1$ and

$$F(\gamma)=(|A|^2|2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|+12|B|^2|[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|+2|AB||6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k-5[(\lambda-\delta)(\beta-\alpha)+1]^{2k}|]b^4+8[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}b(4-b^2)+(6|A||[(\lambda-\delta)(\beta-\alpha)+1]^{2k}-[(\lambda-\delta)(\beta-\alpha)+1]^k$$

$[3(\lambda-\delta)(\beta-\alpha)+1]^k|+2|B||7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|)b^2(4-b^2)\gamma+\{(4-b^2)(4b^2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}+3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k(4-b^2))-8[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}b(4-b^2)\}\gamma^2\cdot$

As $F(\gamma)$ is an increasing function, $Max\,F(\gamma)=F(1)$ is also satisfactorily applicable. Therefore,

$$|a_2a_4-a_3^2| \leq \frac{(A-B)^2}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k}\,g(b), (17)$$

Where $g(b)=F(1)$. So

$g(b)=(|A|^2|2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|+12|B|^2|[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|+2|AB||6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k-5[(\lambda-\delta)(\beta-\alpha)+1]^{2k}|-6|A||[(\lambda-\delta)(\beta-\alpha)+1]^{2k}-[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|-2|B||7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|-4[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}+3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k)b^4+[24|A||[(\lambda-\delta)(\beta-\alpha)+1]^{2k}-[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|+8|B||7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}-6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k|-8(3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k-2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k})]b^2+48[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k.$

Now

$$g'(b) = 4[|A|^2|2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| + 12|B|^2|[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| + 2|AB||6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k - 5[(\lambda-\delta)(\beta-\alpha)+1]^{2k}| - 6|A||[(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| - 2|B||7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| - 4[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} + 3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k]b^3 + 2[24|A||[(\lambda-\delta)(\beta-\alpha)+1]^{2k} - [(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| + 8|B||7[2(\lambda-\delta)(\beta-\alpha)+1]^{2k} - 6[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k| - 8(3[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k - 2[2(\lambda-\delta)(\beta-\alpha)+1]^{2k})]b.$$

Since $g'(b) < 0$ for $b \in [0,2]$, the maximum value of $g(b)$ at $b = 0$ indicates that $Max\ g(b) = g(0)$. Thus,
(7) can be achieved from (17), that is

$$|a_2a_4 - a_3^2| \le \frac{48(A-B)^2[[(\lambda-\delta)(\beta-\alpha)+1]^k[3(\lambda-\delta)(\beta-\alpha)+1]^k]}{192[(\lambda-\delta)(\beta-\alpha)+1]^k[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}[3(\lambda-\delta)(\beta-\alpha)+1]^k} \le \frac{(A-B)^2}{4[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}}$$

For $b_1 = 0, b_2 = 2$, and $b_3 = 0$ the resulting value is sharp.
Based on Theorem (3.1), we obtain the following corollaries:

3.2. Corollary. If $F(z) \in S^*$, then

$$|a_2a_4 - a_3^2| \le \frac{1}{[2(\lambda-\delta)(\beta-\alpha)+1]^{2k}}.$$

This is obtained for A =1 and B = -1.

By applying $k = 0$ in Theorem (3.1), the finding obtained is in line with Singh and Singh stated below **[10]**:

3.3. Corollary. If $f(z) \in S^*(A, B)$, then

$$|a_2 a_4 - a_3^2| \leq \frac{(A-B)^2}{4}.$$

For $A = 1$, $B = -1$ and $k = 0$, Theorem (3.1) gives the following result due to Janteng, Halim and Darus **[11]**.

3.4. Corollary: If $f(z) \in S^*$, then $\left|a_2 a_4 - a_3^2\right| \leq 1$.

4. **Conclusion**:

the study on the second Hankel determinant for a certain class of functions defined by a differential operator provides valuable insights into the geometric properties and analytic behavior of these functions. The analysis of the second Hankel determinant offers a deeper understanding of the interplay between the coefficients of the functions and their corresponding analytic properties. The results obtained in this study contribute to the existing literature on special functions and can be utilized in various mathematical applications. Further research in this area may lead to new discoveries and applications in complex analysis and related fields.

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